12. Extreme Values and Optimization Part 2

In this lecture, we will discuss

- Extreme Values of Functions of Two Variables
  - Local and Absolute Extreme Values of $z = f(x, y)$
  - Critical points
  - Second Derivatives Test
  - Extreme Value Theorem for Functions of Two Variables
- Constrained Optimization: Lagrange Multipliers
  - Method of Lagrange Multipliers

Extreme Values of Functions of Two Variables

Definition Local and Absolute Extreme Values of $z = f(x, y)$

A function of two variables has a local maximum at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some open ball with center (a, b).]

The number $f(a, b)$ is called a local maximum value. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is a local minimum value.

If the inequalities in the above definition hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Minimum and maximum values of a function are called extreme values.

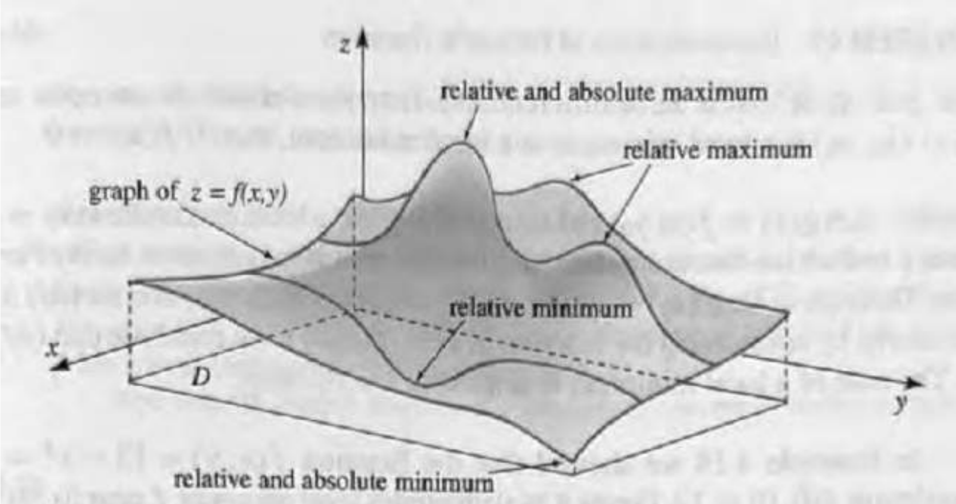


Figure 4.8 Extreme values of a function $z = f(x, y)$.

Theorem Generalization of Fermat's Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

$\nabla f(a,b) = \vec{0}$

Definition Critical Point

A point (a, b) is called a critical point of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Example 1. Find the critical points of the function $f(x, y) = x^2 + y^2 + 2x - 6y + 5$.

ANS: By def. we compute

$$f_x = \frac{\partial f}{\partial x} = 2x+2$$

$$f_y = \frac{\partial f}{\partial y} = 2y-6$$

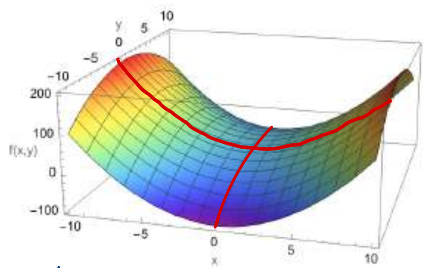
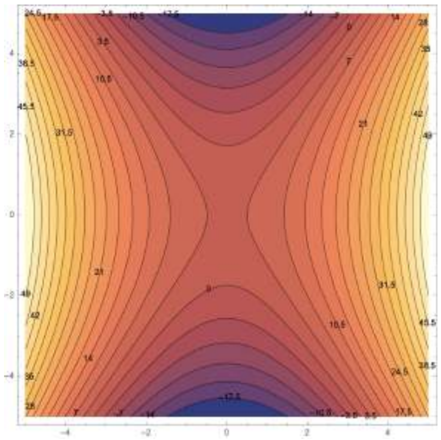
To find the critical point(s) we set

$$\begin{cases} f_x = 2x+2 = 0 \\ f_y = 2y-6 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 3 \end{cases}$$

Thus the critical point for the give function $f(x,y)$ is $(-1, 3)$.

Definition. Saddle Point

A critical point that is neither a local minimum point nor a local maximum point is called a saddle point.

Graph of $f(x,y)=2x^2-y^2+3$	Level (contour) curve of f near $(0,0)$
Exercise: verify $(0,0)$ is a critical point for f.  At $(0,0)$, the graph of f has the shape of a saddle.	

Second Derivatives Test

Suppose the second partial derivatives of f are continuous on an open ball with center (a,b), and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [that is, (a,b) is a critical point of f]. Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$$

(a) If $D>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

(c) If $D<0$, then $f(a,b)$ is not a local maximum or minimum (i.e., it is a saddle point).

Remarks.

1. If $D=0$, the test gives no information: f could have a local maximum or local minimum at (a,b), or (a,b) could be a saddle point of f.
2. To remember the formula for D it's helpful to write it as a determinant:

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=f_{xx}f_{yy}-(f_{xy})^2$$

The following matrix is called Hessian matrix

$$Hf(x,y)=\begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

Example 2. Find the critical point of the function $f(x,y) = 4x - 4y^2 - \ln(|x+y|)$. Then use the Second Derivative Test to examine the critical point.

ANS: Step 1. Find the critical point(s):

We set
$$\begin{cases} f_x = \frac{\partial f}{\partial x} = 4 - \frac{1}{x+y} = 0 \\ f_y = \frac{\partial f}{\partial y} = -8y - \frac{1}{x+y} = 0 \end{cases}$$

To solve this eqn, first note $\frac{1}{x+y} = 4$ from the 1st eqn. Plug it into the 2nd one,

We have
$$-8y - 4 = 0 \Rightarrow y = -\frac{1}{2}$$

Then plug $y = -\frac{1}{2}$ into $\frac{1}{x+y} = 4$ we have.

$$4 = \frac{1}{x - \frac{1}{2}} \Rightarrow x - \frac{1}{2} = \frac{1}{4} \Rightarrow x = \frac{3}{4}$$

Thus the critical point is $(\frac{3}{4}, -\frac{1}{2})$.

Note f_x and f_y are not defined at points $x+y=0$. But we don't need to worry about them because $f(x,y)$ is not defined on $x+y$ since $\ln|x+y|$ appears in $f(x,y)$.

Step 2. We use the second derivative test to classify the critical point.

Recall

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ \underset{\uparrow = f_{xy}}{f_{yx}} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

We need to compute.

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(4 - \frac{1}{x+y}\right) = \frac{1}{(x+y)^2}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(-\left(8y + \frac{1}{x+y}\right)\right) = \frac{1}{(x+y)^2} - 8$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(4 - \frac{1}{x+y}\right) = \frac{1}{(x+y)^2}$$

Thus the discriminant

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = \frac{1}{(x+y)^2}\left(\frac{1}{(x+y)^2} - 8\right) - \frac{1}{(x+y)^4} = -\frac{8}{(x+y)^2}$$

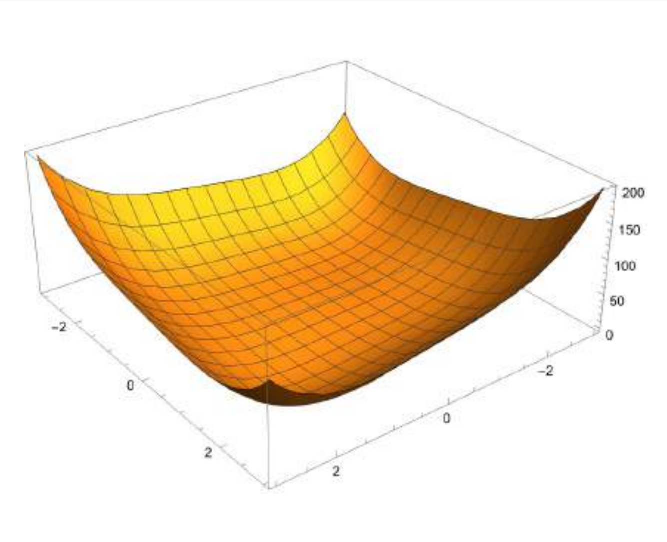
Applying the second derivative test, we have

$$D\left(\frac{3}{4}, -\frac{1}{2}\right) = -\frac{8}{\text{positive number}} < 0$$

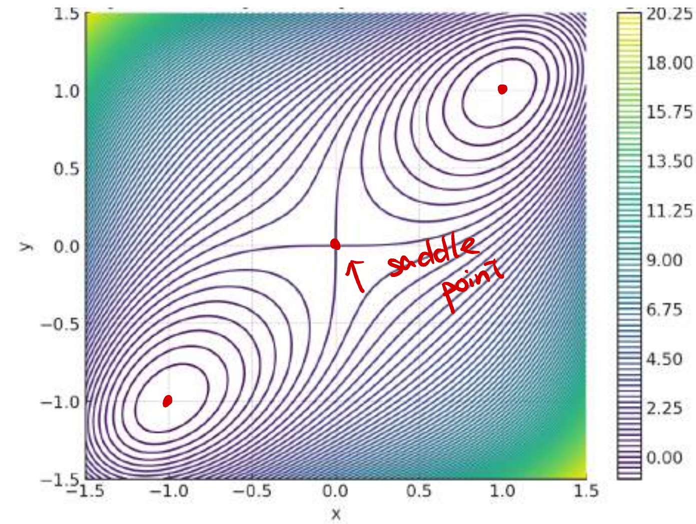
Thus by the 2nd derivative test (case (c)), we know $f\left(\frac{3}{4}, -\frac{1}{2}\right)$ is a saddle point.

Example 3. Find the local maximum and minimum values and saddle points of $f(x,y)=x^4+y^4-4xy+1$.

Graph of $f(x,y)=x^4+y^4-4xy+1$



Level (contour) curve of f



ANS: We first compute the critical points. Let

$$\begin{cases} f_x = \cancel{4}x^3 - \cancel{4}y = 0 \\ f_y = \cancel{4}y^3 - \cancel{4}x = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x^3 - y = 0 \Rightarrow y = x^3 \\ y^3 - x = 0 \end{cases}$$

We plug $y=x^3$ into the 2nd eqn, then

$$x^9 - x = 0 \Rightarrow x(x^8-1) = x(x^4+1)(x^4-1) = x(x^4+1)(x^2+1)(x^2-1) = 0$$

Note x^4+1 and x^2+1 cannot be 0 for any $x \in \mathbb{R}$.

Thus the solutions are $x = 0, 1, -1$.

Then $y = x^3 = 0, 1, -1$.

Therefore the critical points are

$$(0,0),\ (1,1),\ (-1,-1)$$

Next, we compute the 2nd partial derivatives to get $D(x,y)$:

$$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad f_{yy} = 12y^2.$$

Thus

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16$$

Then we examine each critical points:

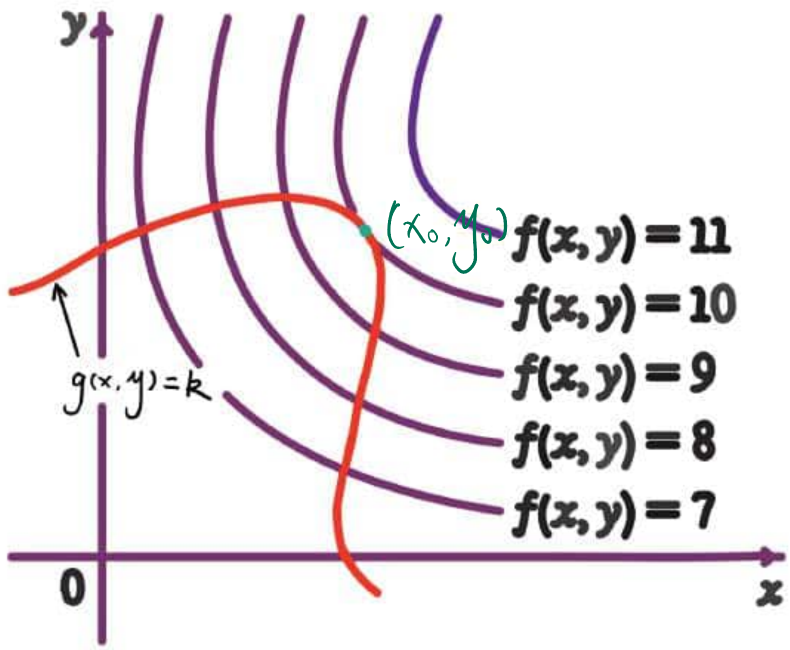
- $(0,0)$: $D(0,0) = -16 < 0$, thus by the second derivative test, $(0,0)$ is a saddle point.
- $(1,1)$; $D(1,1) > 0$ and $f_{xx}(1,1) = 12 > 0$. Thus from case (a) of the second derivative test, $f(1,1) = -1$ is a local minimum
- $(-1,-1)$: $D(-1,-1) > 0$ and $f_{xx}(-1,-1) = 12 > 0$. Thus $f(-1,-1) = -1$ is also a local minimum.

Lagrange Multipliers

Now we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

Let's consider the geometric basis of Lagrange's method for functions of two variables.

- We want to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. That is, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.
- The following figure shows this curve together with several level curves of f.



- These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.
- To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.
- It appears from the figure that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)
- This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

Theorem Lagrange Multipliers for Functions of Two Variables
Let $f, g : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be functions with continuous first derivatives. If the function $f(x, y)$ has a local maximum or a local minimum subject to the constraint $g(x, y) = k$ at $\mathbf{x}_0 = (x_0, y_0)$, and if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, for some real number λ.

Remark. The number in the above theorem is called a **Lagrange multiplier**.

We can generalize the above discussion to functions with 3 variables

Method of Lagrange Multipliers for Functions of Three Variables

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$] :
(a) Find all values of x, y, z, and λ such that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

and
(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example 7.

A company manufactures x units of one item and y units of another. The total cost in dollars, C, of producing these two items is approximated by the function

$$C = 5x^2 + 2xy + 3y^2 + 500.$$

(a) If the production quota for the total number of items (both types combined) is 30 , find the minimum production cost. $g(x,y) = x+y = 30$

(b) Estimate the additional production cost or savings if the production quota is raised to 31 or lowered to 29 .

ANS: We want to minimize $C(x,y)$ subject to

$g(x,y) = x+y = 30$

We compute

$$\nabla C(x,y) = \left(\frac{\partial C}{\partial x}, \frac{\partial C}{\partial y}\right) = (10x+2y, 2x+6y)$$

$$\nabla g(x,y) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (1, 1)$$

Use Lagrange multiplier method.

Step (a). We need to find all x, y, λ such that

$$\begin{cases} \nabla C = \lambda \nabla g \\ g(x,y) = 30 \end{cases} \Rightarrow \begin{cases} (10x+2y, 2x+6y) = \lambda(1,1) \\ x+y = 30 \end{cases}$$

$$\begin{cases} 10x+2y=\lambda \\ 2x+6y=\lambda \\ x+y=30 \rightarrow y=30-x \end{cases}$$

$$\Rightarrow \begin{cases} 10x+2(30-x)=\lambda \\ \qquad\quad \| \qquad\qquad\quad \| \\ 2x+6(30-x)=\lambda \end{cases}$$

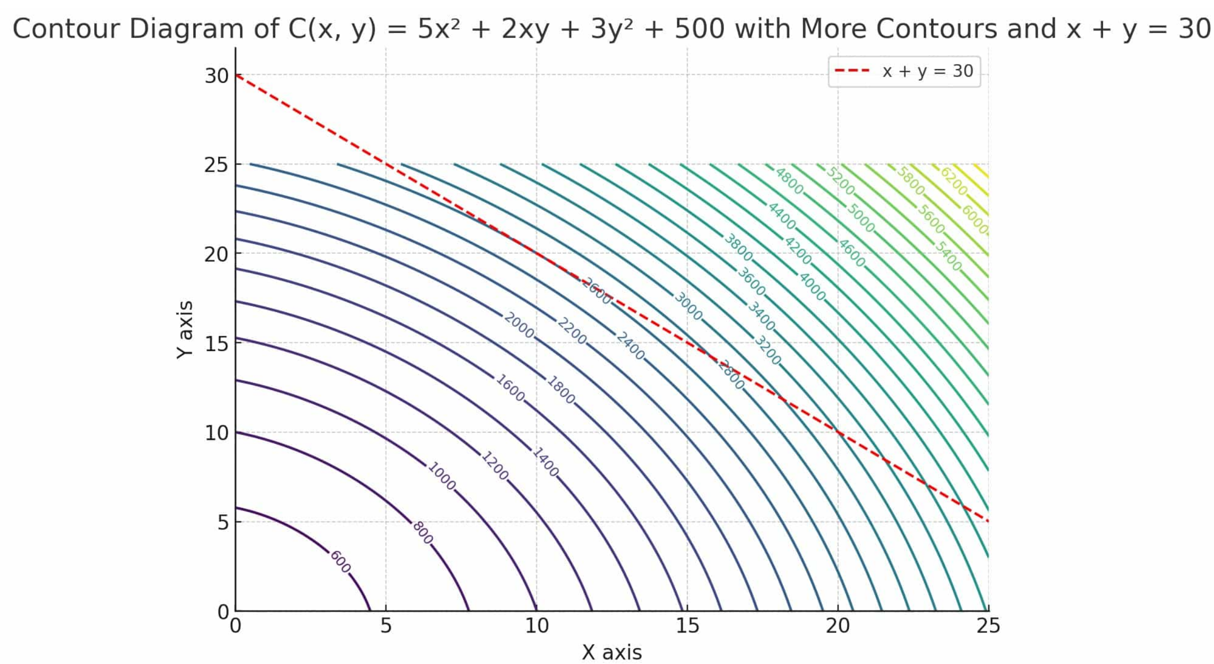
$$\Rightarrow 8x+60=-4x+180 \Rightarrow 12x=120 \Rightarrow x=10$$

Then $y=30-10=20$. and $\lambda = 2x+6y = 140$

$C(10,20) = 5\cdot(10)^2 + 2\cdot(10)\cdot(20) + 3\cdot(20)^2 + 500 = 2600$ dollars.

We can confirm $C(10,20)$ is the min cost by either plug in a different value (say $x=30$, $y=0$) and compare with $C(10,20) = \$2600$.

We can also draw the graph of $C(x,y)$ using software or website.



(b). If $g(x,y)=31$, we have

$$\begin{cases} 10x+2y=\lambda \\ 2x+6y=\lambda \\ x+y=31 \end{cases}$$

Note this part is roughly the same

Thus $\lambda \approx 140$ as before.

this part is the same as (a)

Then $\nabla C(x,y) \approx \lambda \underline{\nabla g(x,y)} = \lambda(1,1)$

Thus $\nabla C(x,y) \approx (\lambda, \lambda)$

Therefore, the rate of change of C is roughly $\lambda = 140$.

Thus increasing production by 1, with cause cost increase by approximately \$140.

Similarly, decreasing production by 1, will save appro. \$140.

Comment. Another method is to solve explicity the eqn.

$$\begin{cases} 10x+2y=\lambda \\ 2x+6y=\lambda \\ x+y=31 \end{cases} \Rightarrow \begin{cases} x=\frac{31}{3} \\ y=\frac{62}{3}, \end{cases}$$

Then compute the corresponding C, and compare with (a).

We will get the cost increase by \$142.33.

Similarly, changing $x+y=29$, C will decrease by \$137.67.

Therefore, our previous estimation is good enough.